Some remarks about Clifford analysis and fractal sets

Stephen Semmes
Rice University

Let n be a positive integer, and let $\mathcal{C}(n)$ be the real *Clifford algebra* with n generators $e_1, \ldots, e_n$. Thus $\mathcal{C}(n)$ is an associative algebra over the real numbers with multiplicative identity element 1 such that

$$e_1^2 = e_2^2 = \cdots = e_n^2 = -1 \tag{1}$$

and

$$e_j\, e_l = -e_l\, e_j \quad \text{when } j \neq l. \tag{2}$$

This reduces to the complex numbers $\mathbf{C}$ when $n = 1$, and to the quaternions $\mathbf{H}$ when $n = 2$. For each n, $\mathcal{C}(n)$ has dimension 2^n as a vector space over the real numbers.

Consider the first-order differential operators $\mathcal{D}_L$, $\mathcal{D}_R$ acting on $\mathcal{C}(n)$-valued functions on $\mathbf{R}^n$ defined by

$$\mathcal{D}_L f = \sum_{j=1}^n e_j \, \frac{\partial f}{\partial x_j} \quad \text{and} \quad \mathcal{D}_R f = \sum_{j=1}^n \frac{\partial f}{\partial x_j} \, e_j. \tag{3}$$

Of course, $\mathcal{D}_L f = \mathcal{D}_R f$ when f is real-valued, since real numbers commute with $e_1, \ldots, e_n$ by definition. Also,

$$\mathcal{D}_L^2 f = \mathcal{D}_R^2 f = -\sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2}. \tag{4}$$

We say that f is left or right *Clifford holomorphic* on an open set $U \subseteq \mathbf{R}^n$ if $\mathcal{D}_L f = 0$ or $\mathcal{D}_R f = 0$ on U, respectively.

There are variants of these notions for functions on $\mathbf{R}^{n+1}$, in which the derivative in the extra dimension does not have a coefficient. However, in analogy with the ∂ and $\overline{\partial}$ operators from complex analysis, one considers pairs of operators on the left and right, with and without an additional minus sign for the derivatives in the other n dimensions. The product of two such operators again reduces to the Laplacian. There are also variants for the quaternions with a derivative in another direction, corresponding to the third imaginary basis vector in $\mathbf{H}$.

For the sake of simplicity, let us restrict our attention to the version of Clifford analysis on $\mathbf{R}^n$ determined by the operators $\mathcal{D}_L$, $\mathcal{D}_R$. If h is a harmonic function on an open set in $\mathbf{R}^n$, then $\mathcal{D}_L h$, $\mathcal{D}_R h$ are left and right Clifford holomorphic on the same open set, respectively. These functions are the same when h is real-valued, and hence both left and right Clifford holomorphic. This is basically the gradient of h, expressed as a Clifford-valued function. As in one complex variable, the components of Clifford holomorphic functions are harmonic.

The sum of two left or two right Clifford holomorphic functions is left or right Clifford holomorphic, respectively. Because of the noncommutativity of the Clifford algebra when $n \ge 2$, the product of two Clifford-holomorphic functions is not necessarily Clifford holomorphic. If $a \in \mathcal{C}(n)$ and f is left Clifford-holomorphic, then $f\,a$ is left Clifford holomorphic, and similarly $a\,f$ is right Clifford holomorphic when f is. The product of a constant and a Clifford holomorphic function in the other order is not necessarily Clifford holomorphic.

These properties of products are also reflected in the identities

$$\mathcal{D}_L(f_1\,f_2) = (\mathcal{D}_L f_1)\,f_2 + \sum_{j=1}^{n} e_j\,f_1\,\frac{\partial f_2}{\partial x_j} \tag{5}$$

and

$$\mathcal{D}_R(f_1\,f_2) = \sum_{j=1}^{n} \frac{\partial f_1}{\partial x_j}\,f_2\,e_j + f_1\,(\mathcal{D}_R f_2). \tag{6}$$

The function

$$E(x) = \frac{\sum_{j=1}^{n} x_j\,e_j}{|x|^n} \tag{7}$$

is left and right Clifford holomorphic for $x \ne 0$ and each n, and in fact is a nonzero real multiple of the fundamental solution for $\mathcal{D}_L$ and $\mathcal{D}_R$. This means that $\mathcal{D}_L E = \mathcal{D}_R E$ is a nonzero real multiple of the delta function at 0 in the sense of distributions. This is because E is a nonzero real multiple of $\mathcal{D}_L h = \mathcal{D}_R h$, where $h(x) = |x|$ when $n = 1$, $h(x) = \log|x|$ when $n = 2$, and $h(x) = |x|^{2-n}$ when $n \ge 3$, and h is a multiple of the fundamental solution for the Laplacian in these dimensions. If f is a continuously-differentiable $\mathcal{C}(n)$-valued function on $\mathbf{R}^n$ with compact support, for instance, then it follows that

$$f(x) = c_n \int_{\mathbf{R}^n} E(x-y)\,\mathcal{D}_L f(y)\,dy = c_n \int_{\mathbf{R}^n} \mathcal{D}_R f(y)\,E(x-y)\,dy \tag{8}$$

for some real number $c_n \ne 0$ depending only on n and every $x \in \mathbf{R}^n$.

As a nice little consequence of this formula, every continuous $\mathcal{C}(n)$-valued function on a compact set $A \subseteq \mathbf{R}^n$ with Lebesgue measure 0 can be approximated uniformly by the restrictions of left or right Clifford holomorphic functions on neighborhoods of A. For such a continuous function can be approximated uniformly first by the restrictions to A of continuously-differentiable functions on $\mathbf{R}^n$ with compact support, and then by left or right Clifford-holomorphic functions on neighborhoods of A using the formula. This does

not work when A has positive Lebesgue measure and finite perimeter, because of conditions on the distributional derivatives of the product of the characteristic function of A and functions that are Clifford holomorphic on A. A classical version of this in one complex variable is described in Exercise 2 at the end of Chapter 20 of [11].

If A is totally disconnected, then every continuous function on A can be approximated uniformly by locally constant functions. In particular, there are "fat Cantor sets" which are totally disconnected and have positive Lebesgue measure.

For any closed set $A \subseteq \mathbf{R}^n$, a continuously-differentiable function on A in the sense of the Whitney extension theorem is a continuous function f on A together with a continuous differential df_x, $x \in A$, with the same properties as an ordinary C^1 function. In the present setting, f would take values in $\mathcal{C}(n)$, and df_x would be a real-linear mapping from $\mathbf{R}^n$ into $\mathcal{C}(n)$. The Whitney extension theorem says that there is an ordinary C^1 function on $\mathbf{R}^n$ with values in $\mathcal{C}(n)$ equal to f on A and whose differential is equal to df_x for $x \in A$.

If Λ is a real-linear mapping from $\mathbf{R}^n$ into $\mathcal{C}(n)$, then $\mathcal{D}_L \Lambda, \mathcal{D}_R \Lambda \in \mathcal{C}(n)$ can be defined in the usual way. If f is a continuously-differentiable function on A with values in $\mathcal{C}(n)$, then this can be applied to df_x to get $\mathcal{D}_L f(x)$ and $\mathcal{D}_R f(x)$ for each $x \in A$. This is equivalent to applying $\mathcal{D}_L$ or $\mathcal{D}_R$ to a C^1 extension of f to $\mathbf{R}^n$ whose differential at $x \in A$ is df_x.

In particular, $\mathcal{D}_L f = 0$ and $\mathcal{D}_R f = 0$ make sense on A. In terms of C^1 extensions on $\mathbf{R}^n$, this means that $\mathcal{D}_L$ or $\mathcal{D}_R$ of the extension tends to 0 as a point approaches A, respectively. More precise rates of vanishing near A can be obtained from stronger regularity.

It is easy to see that a real-linear mapping $\Lambda : \mathbf{R}^n \to \mathcal{C}(n)$ is uniquely determined by its restriction to any hyperplane in $\mathbf{R}^n$ when $\mathcal{D}_L \Lambda = 0$ or $\mathcal{D}_R \Lambda = 0$. Conversely, if λ is a real-linear mapping from a hyperplane in $\mathbf{R}^n$ into $\mathcal{C}(n)$, then there are extensions of λ to real-linear mappings $\Lambda_1, \Lambda_2 : \mathbf{R}^n \to \mathcal{C}(n)$ such that $\mathcal{D}_L \Lambda_1 = 0$, $\mathcal{D}_R \Lambda_2 = 0$.

If A is contained in a nice C^1 hypersurface in $\mathbf{R}^n$ and $f : A \to \mathcal{C}(n)$ is continuously-differentiable, then df_x is not uniquely determined by f on A. For each $x \in A$, df_x can be changed in the normal direction to the tangent space of the hypersurface at x. Thus one can arrange to have $\mathcal{D}_L f = 0$ or $\mathcal{D}_R f = 0$ on A in this case. By contrast, there are plenty of fractal sets which are not flat in this way, so that the differential is uniquely determined by the function on A. Therefore $\mathcal{D}_L f = 0$ or $\mathcal{D}_R f = 0$ can be a significant condition on a fractal set.

On some fractal sets A, there may be a lot of nonconstant continuously-differentiable functions f for which $df_x = 0$ for each $x \in A$, and hence $\mathcal{D}_L f = \mathcal{D}_R f = 0$. This includes locally constant functions on totally disconnected sets, as well as some functions on connected snowflake sets. If every pair of elements of A can be connected by a rectifiable curve, then any continuously-differentiable function f on A with $df_x = 0$ for each $x \in A$ is constant. Sierpinski gaskets and carpets and Menger sponges are examples of such sets. These sets need not be flat in $\mathbf{R}^n$, so that the differential is determined by the function on A.

Even in this type of restricted situation, there are still the usual problems

with products of functions and their $\mathcal{D}_L$ or $\mathcal{D}_R$ derivatives on A. If there are nonconstant functions f with $df = 0$ on A, then they can be treated much like constants.

In one complex variable, holomorphicity can be characterized by expressing the differential of a function by multiplication in the complex numbers. This also makes sense in higher dimensions, using left or right multiplication in the quaternions or a Clifford algebra, and is well known to be too restrictive in that the solutions are affine. For continuously-differentiable functions on a closed set $A \subseteq \mathbf{R}^n$, there are more possibilities. Functions with vanishing differentials trivially have this property, and as an expansion of affine functions one can allow the coefficients to be continuously-differentiable functions with vanishing differentials. If A is contained in a C^1 curve, then one can choose the differentials to be given by left or right multiplication in $\mathcal{C}(n)$.

Suppose that A is a chord-arc curve in $\mathbf{R}^n$, so that the length of an arc on A is bounded by a constant multiple of the distance between its endpoints. For example, A might be the graph of a Lipschitz mapping on a line. One can check that there is a continuously-differentiable function f on A with any prescribed continuous family of differentials df_x. The function is affine when the differentials are constant, and continuous differentials lead to approximately affine functions on small arcs. In particular, df_x may be defined by multiplication on the left or right by a continuous $\mathcal{C}(n)$-valued function on A.

Let us identify $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ with $x_1\, e_1 + \cdots + x_n\, e_n \in \mathcal{C}(n)$. In $\mathcal{C}(n)$,

$$x\, x = -(x_1^2 + \cdots + x_n^2) = -|x|^2. \tag{9}$$

If $x \neq 0$, then it follows that

$$x \left(\frac{-x}{|x|^2} \right) = \left(\frac{-x}{|x|^2} \right) x = 1. \tag{10}$$

Thus x is invertible in $\mathcal{C}(n)$. There are variants of this for quaternions, and in $\mathbf{R}^{n+1}$ where the additional coordinate corresponds to the copy of $\mathbf{R}$ in $\mathcal{C}(n)$.

Let b be a $\mathcal{C}(n)$-valued function on a closed set $A \subseteq \mathbf{R}^n$, and consider

$$(b(x) - b(y))(x - y)^{-1} \text{ or } (x - y)^{-1}(b(x) - b(y)) \tag{11}$$

for $x, y \in A$, $x \neq y$. If b is continuously-differentiable on A with differential given by left or right multiplication in $\mathcal{C}(n)$, then these expressions can be extended continuously to $x = y$, respectively. These expressions are very similar to kernels of commutator operators, but one should be careful about noncommutativity. They certainly are kernels of commutator operators when b is real-valued, and can be viewed as linear combinations of commutators otherwise.

Analogous expressions in the complex plane are very pleasant. If $b(z)$ is a polynomial or a rational function on $\mathbf{C}$, then $(b(z) - b(w))/(z - w)$ is a finite sum of products of functions of z and w individually, which corresponds to an operator of finite rank. If $b(z)$ is holomorphic on some open set, then the ratio extends holomorphically across $z = w$. If $b(x) = \alpha + \beta\, x$ or $\alpha + x\, \beta$ on $\mathbf{R}^n$ for some

$\alpha, \beta \in \mathcal{C}(n)$, then the appropriate quotient is equal to β, but noncommutativity prevents one from going further. For smooth functions b on smooth curves in $\mathbf{R}^n$, one gets smoothness of the quotient in terms of the parameterization of the curve, but this is not quite the same thing.

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